On the Functions Representing Distances and Analogous Functions.

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INTRODUCTION.

Having given the analytical expression for the distance between two points x_1, y_1; x_2, y_2 in the plane, it is a simple matter to prove that the six distances connecting any four points in the plane are connected by one relation, and to find this relation.*

Again, having given the analytical expression for the angle between two planes in space, $a_1 x + b_1 y + z = c_1$, $a_2 x + b_2 y + z = c_2$, we easily find that one relation exists between the six angles made by any four planes in space.

In these examples certain functions of the analytical expressions representing a distance or an angle, namely

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 \quad \text{and} \quad \frac{(a_1 a_2 + b_1 b_2 + 1)^2}{(a_1^2 + b_1^2 + 1)(a_2^2 + b_2^2 + 1)},$$

have this in common, that each is a function of four independent variables occurring in pairs, as x_1, y_1; x_2, y_2, in such a way that the interchange of the constituents of one pair with the corresponding constituents of the other pair leaves the functions unaltered. Moreover, the six functions representing the six distances between four points or the six angles made by four planes are connected by one relation, which, however, is fundamentally different in the two cases.

Other geometric magnitudes might be chosen whose analytical expressions would possess the same characteristic properties, and the question then arises,

* See for example Salmon's "Higher Algebra," 4th Edition, p. 27.

what are the typical forms of all such expressions? This question, slightly extended, will be answered in the following paper, and a few geometrical applications of the solution will be given (§§4–7). Using the coordinates of points of the plane to represent the pairs of variables in the functions sought, we shall formulate this question as follows:

Calling a function of the coordinates of two points of the plane, left unaltered by interchanging the two points, a two-point function, it is desired to find the two-point functions possessing the following property:

The ten two-point functions of the ten pairs of points of any five points are connected by two or more relations independent of the coordinates of the points.

Some of the more general terms and properties of Lie's Continuous Groups, such as *infinitesimal transformations, commutators* ("Klammerausdrücke"), *point-invariants, similarity of groups,* will be used and will be supposed known. The phrase "change of variables," or "transformation of the variables," frequently employed in the following analysis, is understood to mean a change in the coordinate system of the plane of reference. It, therefore, denotes a transformation of the form

$$x_i = \phi(x_i', y_i'), \quad y_i = \psi(x_i', y_i'),$$

ϕ and ψ being the same functions whatever may be the subscript i. We shall, for brevity, write "*the function* (i, j)" in place of "*the two-point function of the points* x_i, y_i; x_j, y_j."

In order to determine the two-point functions having the properties stated above, it is convenient to divide them into three classes, according as the ten two-point functions of five points are connected by just two relations, by three, or by more than three such.

DETERMINATION OF THE TWO-POINT FUNCTIONS.

I.—*The Ten Two-Point Functions of Five Points are Connected by Just Two Relations.*

§1.

Let the five points be x_1, y_1; x_2, y_2; .... ; x_5, y_5, and the two-point functions $f(x_i, y_i;\ x_j, y_j)$; $i, j = 1, 2, 3, 4,$ or 5, $i \neq j$. Only eight of these are independent, according to hypothesis. We can therefore construct, in the usual way, two

independent linear partial differential equations satisfied by all the ten two-point functions. Denoting the partial differential coefficients of any solution f of these equations with respect to x_i and y_i by p_i and q_i respectively, the differential equations may be written

$$\sum_{i=1}^{5}(a_i p_i + b_i q_i) = 0, \quad \sum_{i=1}^{5}(A_i p_i + B_i q_i) = 0, \tag{1}$$

the coefficients a_i, b_i, etc., being functions of the ten variables $x_1, y_1; \ldots; x_5, y_5$.

Two different numbers, i, j, say 1, 2, can now be found among 1, 2, 3, 4, 5, such that the equations (1) can be solved for two of the derivatives p_1, q_1, p_2, q_2. Suppose the derivatives thus selected are p_1, q_1. The equations (1) can then be written in the forms

$$\left.\begin{aligned} p_1 &= \alpha p_2 + \beta q_2 + \sum_{i=3}^{5}(\alpha_i p_i + \beta_i q_i), \\ q_1 &= \gamma p_2 + \delta q_2 + \sum_{i=3}^{5}(\gamma_i p_i + \delta_i q_i). \end{aligned}\right\} \tag{2}$$

As these differential equations are satisfied by all the two-point functions of the five points considered, the equations obtained from them by leaving out the terms $\alpha_5 p_5 + \beta_5 q_5$ and $\gamma_5 p_5 + \delta_5 q_5$ must be satisfied by all the two-point functions of the four points x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4, whatever may be the values of x_5 and y_5.

These last two variables may, or may not, be present in the coefficients of the equations (2) thus abridged. If they are present, we derive more than two independent differential equations that must be satisfied by the functions (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4). These six functions, containing eight variables, can therefore not all be independent. But then it is easily proved that the ten two-point functions of the five points $x_1, y_1; \ldots; x_5, y_5$, satisfy at least three relations, contrary to hypothesis.

Leaving out of (2) in the same manner the terms $\alpha_i p_i + \beta_i q_i$ and $\gamma_i p_i + \delta_i q_i$, $i = 3, 4$, we find that the coefficients α_i, etc., contain the variables $x_1, y_1, x_2, y_2, x_i, y_i$, only.

By definition, any one of the two-point functions is unaltered by interchanging the two points. The ten two-point functions determining the equations

(2) are therefore at most interchanged by interchanging any two of the five points considered. Subjecting the equations (2) to such a change would accordingly give us two new equations equivalent to the first two. It is thus immediately evident that α_i, β_i, etc., are the same functions of $x_1, y_1, x_2, y_2, x_i, y_i$, as α_j, β_j, etc., are of $x_1, y_1, x_2, y_2, x_j, y_j$, and that we could solve the equations (2) for two of the derivatives p_4, q_4, p_5, q_5.

Now, the two-point function $f(x_4, y_4; x_5, y_5)$, satisfying (2), must satisfy the following equations:

$$\left.\begin{aligned} \alpha_4 p_4 + \beta_4 q_4 + \alpha_5 p_5 + \beta_5 q_5 = 0, \\ \gamma_4 p_4 + \delta_4 q_4 + \gamma_5 p_5 + \delta_5 q_5 = 0, \end{aligned}\right\} \tag{3}$$

which are independent for general values of x_1, y_1, x_2 and y_2. Substituting some constant values for these variables that will not destroy the independence of the equations (3), we find that the function (4, 5) must satisfy two independent differential equations of the form

$$\left.\begin{aligned} c_4 p_4 + d_4 q_4 + c_5 p_5 + d_5 q_5 = 0, \\ C_4 p_4 + D_4 q_4 + C_5 p_5 + D_5 q_5 = 0, \end{aligned}\right\} \tag{4}$$

where the coefficients with the subscript 4 are functions of x_4 and y_4 *only*, and those with the subscript 5 the same functions with x_5 and y_5 in place of x_4 and y_4. We may remark that $c_4 D_4 - d_4 C_4 \neq, 0$, otherwise it would follow from (4) that the ten two-point functions, (1, 2), ..., (4, 5) would satisfy the equations $c_i p_i + d_i q_i = 0$, $i = 1, 2, \ldots, 5$, in which case at most five of them could be independent, contrary to hypothesis.

Now, by the theory of partial differential equations, any solution common to the equations (4) must satisfy the equation obtained by forming the "commutator" of these two and putting it equal to zero,

$$\begin{aligned} &\left(c_4 \frac{\partial}{\partial x_4} + d_4 \frac{\partial}{\partial y_4} + c_5 \frac{\partial}{\partial x_5} + d_5 \frac{\partial}{\partial y_5}\right)\left(C_4 p_4 + D_4 q_4 + C_5 p_5 + D_5 q_5\right) \\ &\quad - \left(C_4 \frac{\partial}{\partial x_4} + D_4 \frac{\partial}{\partial y_4} + C_5 \frac{\partial}{\partial x_5} + D_5 \frac{\partial}{\partial y_5}\right)\left(c_4 p_4 + d_4 q_4 + c_5 p_5 + d_5 q_5\right) = 0. \end{aligned} \tag{5}$$

The left-hand member is readily seen to be the commutator of $c_4 p_4 + d_4 q_4$ and $C_4 p_4 + D_4 q_4$ plus the commutator of $c_5 p_5 + d_5 q_5$ and $C_5 p_5 + D_5 q_5$. The new equation is, therefore, of the same form as the equations (4).

New differential equations may now be formed from this one and each of the equations (4), and so on. As all the differential equations so obtained have a solution (4, 5) in common, at most three of them can be independent. If three of them are independent, the ten two-point functions of the five points $x_1, y_1; \ldots; x_5, y_5$ are connected by at least three relations. For, let $\alpha_4' p_4 + \beta_4' q_4 + \alpha_5' p_5 + \beta_5' q_5 = 0$ be any one of the equations satisfied by (4, 5), and it is clear that the differential equation

$$\alpha_1' p_1 + \beta_1' q_1 + \alpha_2' p_2 + \beta_2' q_2 + \alpha_3' p_3 + \beta_3' q_3 + \alpha_4' p_4 + \beta_4' q_4 + \alpha_5' p_5 + \beta_5' q_5 = 0$$

is satisfied by all the ten two-point functions considered. We should have at least three such equations, independent of each other, and, therefore, at most $10 - 3 = 7$ independent common solutions.

Consider next the case where only two of the differential equations of the form

$$\alpha_4' p_4 + \beta_4' q_4 + \alpha_5' p_5 + \beta_5' q_5 = 0$$

are independent. The differential equation (5) should here be derivable from the equations (4). Applying this condition, we find the following identity in p_4 and q_4:

$$\left(c_4 \frac{\partial}{\partial x_4} + d_4 \frac{\partial}{\partial y_4}\right)\left(C_4 p_4 + D_4 q_4\right) - \left(C_4 \frac{\partial}{\partial x_4} + D_4 \frac{\partial}{\partial y_4}\right)\left(c_4 p_4 + d_4 q_4\right)$$
$$\equiv \phi_5 (c_4 p_4 + d_4 q_4) + \psi_5 (C_4 p_4 + D_4 q_4),$$

ϕ_5 and ψ_5 being functions of x_5 and y_5 only. Since none other of the coefficients involve x_5 and y_5, we can take some general constant values of these and obtain the result that the commutator of $c_4 p_4 + d_4 q_4$ and $C_4 p_4 + D_4 q_4$ is of the form

$$m (c_4 p_4 + d_4 q_4) + n (C_4 p_4 + D_4 q_4),$$

where m and n are constants. But this is evidently the necessary and sufficient condition that the two expressions

$$c_4 p_4 + d_4 q_4, \quad C_4 p_4 + D_4 q_4, \tag{6}$$

regarded as *infinitesimal transformations* in Lie's group theory, generate a two-

parametric group of point transformations in two variables. By a transformation of the variables x_4, y_4 of the form

$$x_4 = \phi(x, y), \quad y_4 = \psi(x, y),$$

the two expressions (6) can be changed into one of the following two types:*

$$p, q; \quad p, \quad xp + yq. \tag{7}$$

We have thus the theorem:

1. *If the two-point function* (i, j), $i \neq j$, *is of such a nature that by selecting any five points* $x_1, y_1; \ldots; x_5, y_5$, *the ten functions* (i, j), $i, j = 1, 2, \ldots, 5$; $i \neq j$, *are connected by just two relations independent of the coordinates of these five points, then must the function* (i, j) *be a two-point invariant of a continuous group in two variables similar to one or other of the groups* (7).

Conversely, if a given two-point function is a two-point invariant of such a group, then must the ten two-point functions of any five points be connected by at least two relations, as they satisfy two independent linear partial differential equations.

II.—*The Ten Two-Point Functions of Five Points are Connected by just Three Relations.*

§2.

In this case we have three differential equations corresponding to the equations (1). Two cases may now present themselves as follows:

(A). The three equations can be solved for three of the derivatives p_1, q_1, p_2, q_2.

(B). From the three equations can be eliminated the derivatives p_1, q_1, p_2, q_2.

In (A) we can proceed as above in the case 1. We get three independent equations corresponding to the equations (4), from which we build commutators and get equations of the form (5). The condition that the latter should be derivable from the former gives two different results, found by equating to zero

* See Lie's "Differentialgleichungen mit bekannten infinitesimalen Transformationen," page 425. The types p, xp; q, xq are omitted here, as in our groups the determinant $c_4 D_4 - C_4 d_4 \neq 0$.

the determinants of the coefficients of all the equations obtained. These results are:

1°. The two-point function (4, 5) satisfies a differential equation of the form

$$s_4 p_4 + t_4 q_4 = 0, \tag{8}$$

where s_4 and t_4 are functions of x_4 and y_4 only.

2°. The two-point functions are two-point invariants of a three-parametric continuous group in two variables.

In case 1°, the ten two-point functions of five points must evidently satisfy the five independent differential equations

$$s_i p_i + t_i q_i = 0, \qquad i = 1, 2, \ldots, 5.$$

They must, therefore, satisfy at least five relations, contrary to hypothesis.

In the case 2°, no two of the independent infinitesimal transformations of the group considered can differ only by a factor. For, let $ap + bq$, $\phi(ap + bq)$ be two such transformations. The functions (4, 5) must then satisfy the equations

$$a_4 p_4 + b_4 q_4 + a_5 p_5 + b_5 q_5 = 0, \quad \phi_4(a_4 p_4 + b_4 q_4) + \phi_5(a_5 p_5 + b_5 q_5) = 0,$$

from which we get

$$a_4 p_4 + b_4 q_4 = 0, \quad a_5 p_5 + b_5 q_5 = 0,$$

equations of the form considered in 1°.

The three-parametric groups in two variables having no such pairs of infinitesimal transformations are of the following types:*

$$\left.\begin{array}{l} p, \quad q, \quad xp + cyq, \quad c \neq 0; \\ p + x^2 p + xyq, \quad q + xyp + y^2 q, \quad yp - xq; \\ xq, \quad xp - yq; \quad yp; \\ p, \quad q, \quad xp + (x + y)q. \end{array}\right\} \tag{9}$$

It is readily found that the two-point invariants of any one of these groups can be written in such a form as to satisfy all the requirements of the two-point functions under consideration.

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* "Theorie der Transformationsgruppen," Lie-Engel, Vol. III, p. 57. The types here given are taken from the list on p. 436.

Let us now consider case (B). One of the three differential equations satisfied by the ten functions (1, 2), ... , (4, 5), can be written

$$s_3 p_3 + t_3 q_3 + s_4 p_4 + t_4 q_4 + s_5 p_5 + t_5 q_5 = 0.$$

Suppose $s_4 \neq 0$. The two functions (1, 4) and (2, 4) must evidently satisfy the differential equation

$$p_4 + \frac{t_4}{s_4} q_4 = 0. \tag{10}$$

It follows that $\frac{t_4}{s_4}$ is a function of x_4, y_4 only, unless the two-point functions are constants merely. If, however, $\frac{t_4}{s_4}$ is a function of x_4, y_4 only, the equation (10) is of the same form as (8) in $1°$, an impossible case.

To resume, we have the theorem:

2. *The necessary and sufficient condition that the ten two-point functions of any five points satisfy just three relations is that they are two-point invariants of a three-parametric continuous group in two variables similar to one or other of the types* (9).

By examining the invariants of the groups considered we find that *the six two-point functions of the kind here defined in any four points satisfy just one relation.*

III.—*The Ten Two-Point Functions of any Five Points Satisfy at least Four Relations.*

§3.

In the first place we find, by considering the possible sets of four relations connecting the ten two-point functions (1, 2), ... , (4, 5), that one relation at least must connect the six functions (1, 2), ... , (3, 4) of any four points. Then we find that one relation at least must exist between the functions (1, 2), (1, 3), (2, 3); (1, 4), (2, 4); (1, 5), (2, 5). If (2, 5) is actually present in this relation, we may replace x_1, y_1, x_3, y_3, x_4, y_4 by arbitrary constants, and change the variables so that (1, 5) becomes the new x_5. We thus obtain (2, 5) as a function of x_5, x_2, y_2, which must evidently reduce to a function of x_5, x_2 only; say $(2, 5) \equiv f(x_2, x_5)$. The ten two-point functions of any five points will obviously satisfy at least five relations.

In the same manner all possible cases of having one relation connect the functions (1, 2), (1, 3), (2, 3); (1, 4), (2, 4); (1, 5), (2, 5) may be considered. The same result will be obtained.

If more than five relations connect the ten two-point functions of five points, we find that one relation at least connects the three two-point functions of three points. Suppose there is one such relation. Let the three functions be

$$f(x_1, x_2), \quad f(x_2, x_3), \quad f(x_3, x_1),$$

and, by our usual process of building a linear partial differential equation, etc, we find that this can, by a suitable choice of variables, be put into the form

$$p_1 + p_2 + p_3 = 0.$$

The two-point function (1, 2) is, accordingly, of the type $(x_1 - x_2)^2$. Additional relations existing among the two-point functions considered will cause these to be constants merely, as is readily proved.

Accordingly, we have the theorem:

3. *If at least four relations connect the ten two-point functions of any five points, these two-point functions are, by a proper choice of variables, reducible to the form*

$$f(x_i, x_j).$$

Conversely, if the two-point functions have this form, at least five relations connect the ten two-point functions of five points.

More than five relations connecting these functions could exist only if the two-point functions, which are not supposed to be constants merely, are reducible to the type

$$(x_i - x_j)^2.$$

By consulting the theorems 1–3, observing that each of the groups (9) contains two-parametric subgroups similar to one or other of the groups (7), we can make the following statement:

4. *The necessary and sufficient condition that the ten two-point functions of any five points are connected by two or more relations, is that the two-point function* (i, j), $i \neq j$, *is reducible to the form* $f(x_i, x_j)$ *by a change of variables, or that the two-point functions are two-point invariants of a continuous group in two variables similar to one or other of the groups*

$$p, q; \quad p, xp + yq.$$

Thus, by a proper choice of the variables x, y, the two-point functions possessing the property just stated are represented by the three different types

$$f(x_i, x_j); \quad f(x_i - x_j, y_i - y_j); \quad f\left(\frac{x_i - x_j}{y_i}, \frac{x_i - x_j}{y_j}\right).$$

SOME GEOMETRICAL APPLICATIONS.

I.—*Concerning Angles.*

§4.

Certain functions of the angles

$$\cos^{-1} \frac{a_1 a_2 + b_1 b_2 + 1}{\sqrt{1 + a_1^2 + b_1^2}\sqrt{1 + a_2^2 + b_2^2}} \quad \text{and} \quad \tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}$$

between the two planes $a_1 x + b_1 y + z = c_1$, $a_2 x + b_2 y + z = c_2$ and the two straight lines $y + m_1 x = b_1$, $y + m_2 x = b_2$ respectively, as

$$\frac{(a_1 a_2 + b_1 b_2 + 1)^2}{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)} \quad \text{and} \quad \frac{(m_1 - m_2)^2}{(1 + m_1 m_2)^2},$$

are types of two-point functions.

The first, written in the variables x, y in place of a, b, is a two-point invariant of the second of the groups (9). The six angles made by four planes in space are connected by one relation.

The second may be written $[\tan(\tan^{-1} m_1 - \tan^{-1} m_2)]^2$, and is one of the types of two-point functions considered in §3. The three angles made by three straight lines in a plane are connected by one relation.

II.—*Concerning Distances.*

§5.

The two-point functions which are such that just one relation exists between the six two-point functions of four points must, as has been demonstrated, be two-point invariants of continuous groups similar to the group-types (9). Replacing the first of these types by the similar group

$$p, \quad q, \quad yp - xq + k(xp + yq),$$

the two-point invariants of the two points x_1, y_1; x_2, y_2 are respectively for the four groups

$$\left.\begin{array}{ll}
\text{A.} & \{(x_2-x_1)^2+(y_2-y_1)^2\}\, e^{2k\tan^{-1}\frac{y_2-y_1}{x_2-x_1}}; \\
\text{B.} & \dfrac{(x_2-x_1)^2+(y_2-y_1)^2+(x_1y_2-x_2y_1)^2}{(1+x_1x_2+y_1y_2)^2}; \\
\text{C.} & (x_1y_2-x_2y_1)^2; \\
\text{D.} & (x_2-x_1)^2\, e^{-2\frac{y_2-y_1}{x_2-x_1}}.
\end{array}\right\} \qquad (11)$$

Denoting, as above, by (a, b), the two-point function of the two points x_a, y_a; x_b, y_b, the relations connecting $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$, for the different types given, are found by elimination of the coordinates x_1, y_1, etc. They are, respectively,

A. The result of eliminating s and t between the equations

$$[23](s-1)^c-[13]\,s^c+[12]=0,$$
$$[24]\,(t-1)^c-[14]\,t^c+[12]=0,$$
$$[34]\,(t-s)^c-[14]\,t^c+[13]\,s^c=0,$$

where
$$c=\frac{k-i}{k+i},\ i=\sqrt{-1};\ \text{and } [ab]=(a, b)^{\frac{1-c}{2}};$$

B. $$1+2[23][24][34]=[23]^2+[24]^2+[34]^2,$$

where
$$[ab]=\frac{\overline{(a, b)}-\overline{(1, a)}\,\overline{(1, b)}}{\{(\overline{(1, a)}^2-1)(\overline{(1, b)}^2-1)\}^{\frac{1}{2}}},$$

and
$$\overline{(a, b)}=\frac{1}{\sqrt{(a, b)+1}}=\frac{1+x_1x_2+y_1y_2}{\sqrt{1+x_1^2+y_1^2}\,\sqrt{1+x_2^2+y_2^2}}; \qquad (12)$$

C. $$[12][34]+[13][42]+[14][23]=0,$$

where $$[ab]=(a, b)^{\frac{1}{2}};$$

D. The result of eliminating s and t between the equations

$$(s-1)\log(s-1)-s\log s=(s-1)[23]-s[13]+[12],$$
$$(t-1)\log(t-1)-t\log t=(t-1)[24]-t\,[14]+[12],$$
$$(t-s)\log(t-s)-t\log t+s\log s$$
$$=(t-s)[34]-t\,[14]+s\,[13],$$

where $$[ab]=\tfrac{1}{2}\log(a, b).$$

The distance between two points in the plane is a two-point function of the kind here considered. Thus we see that the property of the analytic expression for the distance between two points of remaining unaltered by exchanging the coordinates of the two points, added to the property that the six distances connecting any four points are connected by just one relation, limits this expression to a function of one of the four types (11).

Examining more closely the two-point invariants (11) and the relations (12) we get the following result:*

The distance between two points, or a function of this distance, in the euclidean or non-euclidean plane, is determined by the following conditions:

The distance is a two-point function, which is real if real coordinates are used. Through any general point in the plane no real curve passes, the distance from every point of which to the given point is indeterminate, a constant, or is infinitely great. The six distances connecting any four points in the plane are connected by just one relation, which is an algebraic relation between certain functions of these distances.

To apply these conditions we have first to find what values of the arbitrary constant k and what transformations of the coordinates involved will render certain functions of the invariants (11) real quantities for real coordinates. To answer this question it is plainly sufficient to find all the *real group types*† similar to the types (9).

These types are given in Lie's "Theorie der Transformationsgruppen," Vol. III, p. 436. To the four given in (9), assumed to contain real elements, must be added the following two:

$$p, \quad q, \quad yp - xq + k(xp + yq);$$
$$p - x(xp + yq), \quad q - y(xp + yq), \quad yp - xq;$$

also containing real elements. These groups are similar to the first two of the groups (9), and their two-point invariants (1, 2) are respectively (A) of (11) and

$$\text{E.} \quad \frac{(x_1 y_2 - x_2 y_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2}{(1 - x_1 x_2 - y_1 y_2)^2}.$$

The two-point invariant (1, 2) of the first of the groups (9) is

$$\text{F.} \quad (y_2 - y_1)^2 (x_2 - x_1)^{-2c}.$$

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* Compare "On the determination of the distance between two points in space of n dimensions," by the author, *Transactions of the Am. Math. Soc.*, Oct., 1903, p. 467.

† Cf. "Theorie der Transformationsgruppen," Vol. III, Chapter 19, for real groups.

Applying the second condition given above to the six invariants obtained, we find that the invariant (F) just given and the two last, (C) and (D) of (11), must be excluded having respectively the curves

$$y = \text{constant}, \quad \frac{y}{x} = \text{constant}, \quad x = \text{constant},$$

of the kind excluded.

The types (A), (B) and (E) must now be considered. The two last fulfill all the conditions given, and are, in fact, functions ($\tan^2 d$ and $\tan^2 id$) of the distance d between two points in the elliptic and hyperbolic plane respectively, proper coordinates being chosen. In the type (A), k must be real by the first condition, and by referring to the relations connecting the six invariants (1, 2), etc., given under (A) in (12), we find that c must be real in order that the relation connecting these six invariants may be algebraic. Since $c = \frac{k-i}{k+i}$, $i = \sqrt{-1}$, from which $k = i\,\frac{1+c}{1-c}$, the only permissible value for k is zero. (If $k = \infty$, the type (A) of (11) will reduce to a particular case of type (F).)

§6.

III.—*Problem: What are the surfaces for which, any five points be taken and joined by chords, the lengths of the ten chords so obtained are connected by two or more relations independent of the location of the points on the surface?*

Let such a surface be given by $z = f(x, y)$, and let us write p, q, r, s, t for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$, as is customary. To avoid confusion, we shall henceforth write $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for p, q in the infinitesimal transformations concerned, reserving the letters p, q for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

The square of the distance connecting the points x_1, y_1, z_1 and x_2, y_2, z_2 is

$$(1, 2) \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

where $$z_1 = f(x_1, y_1), \quad z_2 = f(x_2, y_2).$$

According to the theorem (4), §3, the two-point function (1, 2) must be a two-point invariant of a continuous group similar to one or other of the groups

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}; \quad \frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y};$$

or it must, by a change of variables, be reducible to the form $\phi(x_1, x_2)$.

If the latter is the case, we must have the identity

$$\phi(\alpha_1, \alpha_2) \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

where α_1 is a function of x_1 and y_1, only, α_2 the same function of x_2 and y_2.

Differentiating this equation in turn by x_1 and y_1, and eliminating $\frac{\partial \phi}{\partial \alpha_1}$, we obtain the equation

$$(z_2 q_1 + y_2 - z_1 q_1 - y_1)\frac{\partial \alpha_1}{\partial x_1} + (-z_2 p_1 - x_2 + z_1 p_1 + x_1)\frac{\partial \alpha_1}{\partial y_1} = 0,$$

where $p_1 = \frac{\partial z_1}{\partial x_1}$, etc.

If we now give to x_1 and y_1 general constant values, we get the equation $z_2 a + y_2 b + x_2 c + d = 0$, a, b, etc., being constants, at least one of which is different from zero, as $\frac{\partial \alpha_1}{\partial x_1}$ and $\frac{\partial \alpha_1}{\partial y_1}$ cannot both be zero identically. The surface is, in such a case, a plane, for which the conditions of the problem are, a priori, satisfied.

Thus it remains for us to find the surfaces $z = f(x, y)$ for which the function (1, 2) is a two-point invariant of a group similar to one or other of the groups

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}; \quad \frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Let the infinitesimal transformations of such a group be

$$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}, \quad \gamma\frac{\partial f}{\partial x} + \delta\frac{\partial f}{\partial y}.$$

In order that the function (1, 2) be a two-point invariant of this group, it should satisfy the differential equations

$$\alpha_1\frac{\partial f}{\partial x_1} + \beta\frac{\partial f}{\partial y_1} + \alpha_2\frac{\partial f}{\partial x_2} + \beta_2\frac{\partial f}{\partial y_2} = 0, \quad \gamma_1\frac{\partial f}{\partial x_1} + \text{etc.} = 0,$$

α_1 being written in the variables x_1, y_1; α_2 in x_2, y_2, etc.

That the first of these equations may be satisfied by (1, 2), we must have identically

$$(x_1 - x_2)(\alpha_1 - \alpha_2) + (y_1 - y_2)(\beta_1 - \beta_2) \\ + (z_1 - z_2)(\alpha_1 p_1 + \beta_1 q_1 - \alpha_2 p_2 - \beta_2 q_2) \equiv 0 . \qquad (13)$$

By substituting in turn different sets of constant values for x_2 and y_2 in this equation, we can solve for $\alpha_1 p_1 + \beta_1 q_1$, α_1 and β_1 linearly in terms of x_1, y_1 and z_1, unless the surface sought is a plane.

Disregarding the latter case (an obvious solution of the problem), we find the identity (13) satisfied by

$$\alpha = -az + ky + g, \quad \beta = -bz - kx + h, \quad \alpha p + \beta q = ax + by + c,$$

a, b, c, k, g, h being constants.

Similarly,

$$\gamma = -Az + Ky + G, \quad \delta = -Bz - Kx + H, \quad \gamma p + \delta q = Ax + By + C.$$

The equations

$$\alpha p + \beta q = ax + by + c, \quad \gamma p + \delta q = Ax + By + C$$

(being made consistent by modifying the constants a, b, etc.), will now determine z. Restricting ourselves to real solutions, the following surfaces only satisfy the conditions of the problem under consideration:

1°. Any series of a finite number of parallel planes.
2°. Any series of a finite number of concentric spheres.
3°. Any series of a finite number of co-axial right circular cylinders.

The mutual distances connecting *four* points on either of the first two surfaces are found without much difficulty to be connected by one relation, whereas, in the remaining surface, the mutual distances connecting five points are bound by just two relations. Thus, *the surface consisting of any series of co-axial right circular cylinders is the only real surface for which any five points being taken, the mutual distances (along chords) connecting these five points are connected by just two relations.*

§7.

IV.—*Problem: What are the surfaces for which, any five points being taken and joined by geodesics, the ten geodesic distances so obtained are connected by two or more relations independent of the positions of the points?*

Before going into the details of this problem, we may remark that the six mutual geodesic distances of four points on a surface of constant curvature are connected by one relation.

Any two such surfaces with the same curvature are, as is well known, applicable one upon the other. As representative surfaces, the sphere $x^2+y^2+z^2=a_2$ with curvature $\frac{1}{a^2}$ and the pseudosphere*

$$x^2+y^2=a^2\sin^2\phi,\quad z=a\left(\log\tan\frac{\phi}{2}+\cos\phi\right)$$

with curvature $-\frac{1}{a^2}$ may therefore be selected. The geodesic distance d between the two points x_1, y_1, z_1; x_2, y_2, z_2 are respectively given by

$$\sin^2\frac{d}{2a}=\frac{(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2}{-4\beta_1\beta_2},\quad \sin^2\frac{d}{2ia}=\frac{(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2}{-4\beta_1\beta_2},$$

where α and β are certain functions of x, y and z.†

Now, the distances connecting four points on a sphere are, as we know, connected by one relation. The distances on the pseudosphere in question being of the same type as the distances on the sphere, differing from these only by the multiplier i after a proper choice of coordinates, it is evident that one relation must connect the six mutual distances of four points on the pseudosphere.

Now, we shall prove that the surfaces required in the present problem, if real, must have constant curvature.

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* Darboux, "Theorie des Surfaces," T. III, p. 394.

† Darboux, "Theorie des Surfaces," T. III, p. 401. The variables x, y used by M. Darboux in the formulæ referred to are here replaced by α and β respectively.

Under the conditions of the problem, the distance (1, 2) must be a two-point invariant of a continuous group similar to one of the groups

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}; \qquad \frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \tag{14}$$

or it must be reducible to the form $\phi(x_1, x_2)$ by a change of variables (theorem 4, §3).

For a real surface, the distance (1, 2) could not be of the form $\phi(x_1, x_2)$. This would, namely, require the linear element ds of the surface to be rational in dx, as is easily seen. The linear element of a real surface $z = f(x, y)$,

$$ds = \sqrt{\{(1+p^2)\,dx^2 + 2pq\,dx\,dy + (1+q^2)\,dy^2\}}$$

could, however, not become rational in dx and dy by any change of variables.

Thus the distance (1, 2) must be a two-point invariant of a group similar to one or other of the groups (14). Taking the two points indefinitely near each other, and writing x, y, $x + dx$, $y + dy$ for x_1, y_1, x_2, y_2 respectively, we have

$$(1, 2)^2 = ds^2 = (1+p^2)\,dx^2 + 2pq\,dx\,dy + (1+q^2)\,dy^2.$$

Introducing in this expression the variables x, y used in the groups (14) we obtain

$$ds^2 = E dx^2 + 2F\,dx\,dy + G dy^2,$$

E, F, G being functions of x and y. In order that this expression may be unaltered by the point transformation determined by one or other of the groups (14), ds^2 must satisfy one or other of the following systems of simultaneous partial differential equations in x, y, dx, dy:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,$$
$$\frac{\partial f}{\partial x} = 0, \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + dx\frac{\partial f}{\partial\, dx} + dy\frac{\partial f}{\partial\, dy} = 0,$$

The solutions are respectively

$$a dx^2 + 2b dx\,dy + c dy^2, \quad \frac{1}{y^2}\left(a dx^2 + 2b dx\,dy + c dy^2\right),$$

a, b and c being constants.

Linear transformations will reduce these expressions to the forms

$$dx^2 + dy^2, \quad -\frac{dx^2 + dy^2}{ky^2},$$

which are representative forms for the linear element of surfaces of constant curvature, zero and k respectively. Hence,

If two relations exist between the ten mutual geodesic distances of any five points on a real surface, independent of the coordinates of the five points, this must be a surface of constant curvature. On such a surface one relation exists between the six mutual geodesic distances of any four points.

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